

Invariant Measures and Convergence for Cellular Automaton 184 and Related Processes.

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Abstract. For a class of one-dimensional cellular automata, we review and complete the characterization of the invariant measures (in particular, all invariant phase separation measures), the rate of convergence to equilibrium, and the derivation of the hydrodynamic limit. The most widely known representatives of this class of automata are: Automaton 184 from the classification of S. Wolfram [W1], an annihilating particle system and a surface growth model.

Key words and phrases: Cellular automata, automaton 184, ballistic annihilation, annihilating deterministic motions, surface growth, invariant measures, hydrodynamic limits, rate of convergence to equilibrium.

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1. INTRODUCTION

Cellular Automaton 184 (CA 184) is a discrete time process with state space $\{0, 1\}^{\mathbf{Z}}$ and the following evolution rule: if $\eta \in \{0, 1\}^{\mathbf{Z}}$ is the state at time n then the state η' at time $n + 1$ is defined by

$$\eta'(x) := \begin{cases} 1, & \text{if } \eta(x) = \eta(x+1) = 1 \\ 1, & \text{if } \eta(x) = 1 - \eta(x-1) = 0 \\ 0 & \text{otherwise} \end{cases} \quad \forall x \in \mathbf{Z} \quad (1.1)$$

Here $\eta(x)$ denotes the value of $\eta : \mathbf{Z} \rightarrow \{0, 1\}$ at the coordinate x . The dynamics of CA 184 will be denoted by the operator $C : \{0, 1\}^{\mathbf{Z}} \rightarrow \{0, 1\}^{\mathbf{Z}}$ defined by $C\eta = \eta'$. Thus, formally, CA 184 refers to a sequence $\{\eta_n, n \in \mathbf{N}\}$ such that $\eta_{n+1} = C\eta_n$, $\forall n$.

CA 184 models deterministic motions of identical particles on \mathbf{Z} that obey the following rules: there may be at most one particle per site, and at each integer time each particle inquires whether the site of \mathbf{Z} to the right of its current position is empty of another particle, and if it is so then it instantaneously jumps to this site. These rules follow immediately from (1.1), if one interprets $\eta(x) = 1/0$ as “presence/absence of a particle at the site $x \in \mathbf{Z}$ in η ”.

The number “184” in the name of this process is due to the classification of Wolfram [W1] (see also [W2]) of a class of cellular automata. CA 184 has been used to model traffic ([N], [Nl] and references therein) and to classify densities in binary strings ([CST]). CA 184 has two “stochastic” counterparts. One is the Totally Asymmetric Simple Exclusion Process (TASEP). The second one has the same evolution rule as CA 184 but with “a noise” that is introduced by setting that each particle that can jump will do so with probability

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p independently of anything else; we thus call this process “CA 184 with noise” (CA&N). The invariant measures for TASEP and CA&N have been characterized in [L] and [Y], respectively (curiously, only the case $p \leq 1/2$ was studied in [Y], but we believe that a similar technique allows to extend the results to any p) but the approach employed there (which is a stochastic coupling) does not apply to CA 184 (because of the lack of stochasticity in its dynamics).

Annihilating particle system studied here can be also found in the literature under the names Ballistic Annihilation (BA) and Annihilating Deterministic Motions. We shall adopt here the name BA. BA is a discrete time process with state space $\{-1, 0, 1\}^{\mathbf{Z}}$ and the following dynamics: if ζ is the state at time n then the state ζ' at time $n + 1$ satisfies

$$\zeta'(x) = \begin{cases} 1, & \text{if } \zeta(x-1) = 1 \text{ and neither } \zeta(x) = -1 \text{ nor both } \zeta(x) = 0, \zeta(x+1) = -1 \\ -1, & \text{if } \zeta(x+1) = -1 \text{ and neither } \zeta(x) = 1 \text{ nor both } \zeta(x) = 0, \zeta(x-1) = 1 \\ 0, & \text{otherwise} \end{cases} \quad (1.2)$$

for all $x \in \mathbf{Z}$. Here $\zeta(x)$ denotes the value of $\zeta : \mathbf{Z} \rightarrow \{-1, 0, 1\}$ at the coordinate x . The dynamics of BA will be denoted by the operator $A : \{-1, 0, 1\}^{\mathbf{Z}} \rightarrow \{-1, 0, 1\}^{\mathbf{Z}}$ defined by $A\zeta = \zeta'$. Thus, formally, BA refers to a sequence $\{\zeta_n, n \in \mathbf{N}\}$ such that $\zeta_{n+1} = A\zeta_n$, $\forall n$.

BA may be also interpreted in terms of particles. We shall call them A-particles in order to distinguish them from those that move in CA 184. The values $0, 1, -1$ of $\zeta(x)$ are interpreted by saying that the site x is respectively, free of an A-particle, contains an A-particle with velocity 1, and contains an A-particle with velocity -1 . In the terms of particles, the dynamics of BA acquires the following interpretation: each A-particle moves along \mathbf{R} with its velocity (going in the direction to $-\infty$ ($+\infty$), if the velocity is negative (positive, resp.)) and annihilates when meets another A-particle; upon annihilation both A-particles disappear from the system forever. To link this interpretation with (1.2), it is necessary to note that if A-particles occupied only the sites of \mathbf{Z} then after a unit of time has elapsed the survived A-particles are again solely at the sites of \mathbf{Z} .

BA is a natural model for the reaction $A + B \rightarrow \text{inert}$. For an extensive list of references in this respect, we refer a reader to [EF]. A modification of BA in which the set of possible particle velocities is larger than $\{-1, 1\}$ has been considered in the works [BRL], [KSL]. We also mention here the work [E] which studies a coalescing particle system employing a very simple relation between it and BA. This coalescing particle system is a process in which particles move as in BA but coalesce instead of annihilate, upon collision; coalesced particles choose then a new velocity from $\{-1, 1\}$ with equal probabilities.

Surface Growth Model (SG) is a discrete time process with state space \mathcal{R} , the space of piecewise linear functions (from \mathbf{R} to \mathbf{R}) that have slope either $+1$, -1 , or 0 between any two consequent integer abscissas, and the following dynamics: If $f(\cdot)$ is the state at time n then $\hat{f}(\cdot) \in \mathcal{R}$, the state at time $n+1$, is obtained from $f(\cdot)$ by the following rule: if $x \in \mathbf{Z}$ is a local minimum of f , then the values of \hat{f} at $[x-1, x+1]$ are obtained by reflecting the graph of f around the straight line which links the points $(x-1, f(x-1))$ and $(x+1, f(x+1))$; at any point $y \in \mathbf{R}$ which does not have a local minimum of f in its neighborhood of radius 1, we set $\hat{f}(y) = f(y)$. SG will be the name of a sequence $\{f_n(\cdot), n \in \mathbf{N}\}$ such that $f_{n+1}(\cdot) = \hat{f}_n(\cdot)$, $\forall n$.

Imagine $f(\cdot)$ as the surface of a two-dimensional solid above some reference horizontal line. Imagine then that diamond shaped particles of the side length 1 are thrown on this solid, and those of them that fall in local minima of $f(\cdot)$ stick to the solid while others disappear. The surface of the new solid will be then what we have defined as $\hat{f}(\cdot)$. This justifies the name of the process just introduced. We observe that it is also known under the name polynuclear growth model (PNG) and (one of its modifications) was first studied almost 30 years ago in [Ber] (see also [KS2] for a list of more recent works).

Equivalence of dynamics of two processes means here an existence of a transformation T from the state space of one process to that of another one such that if $T\eta = \zeta$ then $T\eta' = \zeta'$, where η' and ζ' are the states of these processes at time $n + 1$ given their states at time n were η and ζ respectively. The fact that the dynamics of CA 184, BA and SG are equivalent has been known since at least the work [KS]. We present it in Assertions 1.1 and 1.2 below. The proofs are straightforward and thus, omitted.

Assertion 1.1. (equivalence between CA 184 and BA) Define $T_{184,BA} : \{0,1\}^{\mathbf{Z}} \rightarrow \{-1,0,1\}^{\mathbf{Z}}$ by

$$(T_{184,BA}\eta)(i) = 1 - \eta(i) - \eta(i+1), \quad i \in \mathbf{Z} \quad (1.3)$$

Then, if for some η and ζ , $T_{184,BA}(\eta) = \zeta$ then $T_{184,BA}(\eta') = \zeta'$, where η' relates to η via (1.1), and ζ' relates to ζ via (1.2).

Remark 1.1. $T_{184,BA}$ is an injection but not a surjection. Indeed, if any two consequent sites of \mathbf{Z} are occupied by A-particles with opposite velocities in a configuration $\zeta \in \{-1,0,1\}^{\mathbf{Z}}$ then there does not exist a $\eta \in \{0,1\}^{\mathbf{Z}}$ such that $T_{184,BA}\eta = \zeta$. As a consequence, a certain care is necessary, when CA 184 is studied with the help of BA (the reader will see it in the proofs of Theorems 2.2 and 2.3).

Assertion 1.2. (equivalence between BA and SG) Define $T_{BA,SG} : \{-1,0,1\}^{\mathbf{Z}} \rightarrow \mathcal{R}$ of the following form

$$\begin{aligned} (T_{BA,SG}\zeta)(n) - (T_{BA,SG}\zeta)(n-1) &= \zeta(n), \quad n \in \mathbf{Z} \\ (T_{BA,SG}\zeta)(0) &= 0 \end{aligned} \quad (1.4)$$

(certainly, it is sufficient to determine $f \in \mathcal{R}$ on \mathbf{Z}). Then, if $T_{BA,SG}\zeta = f + \alpha$ for some $\zeta \in \{-1,0,1\}^{\mathbf{Z}}$, $f \in \mathcal{R}$ and $\alpha \in \mathbf{R}$, then $T_{BA,SG}\zeta' = \hat{f} + \beta$, where ζ' relates to ζ via (1.2), \hat{f} relates to f via the dynamics of SG and β is a real number whose value depends on α and f . (Above, the relation $g = h + \alpha$ for functions g and h and a number α means $g(x) = h(x) + \alpha \forall x \in \mathbf{R}$.)

Remark 1.2. The actual values of the constants α and β are ignored because they will be of no importance, when SG is employed to investigate BA and CA 184, and also when BA or CA 184 is employed to investigate phenomena related to modification of the shape of the surface in SG. In this respect we observe that in SG, the surface both grows and changes its shape, but only the phenomena related to the shape change will be of interest to us.

The contents of this paper: Section 2 characterizes the invariant measures for the dynamics of the processes defined above. This is done with a help of a quite elementary

property of BA presented in Assertion 2.1; it states at any time n the distance between two diverging subsequent particles (i.e., no any other particle in between) cannot be less than $2n + 1$. (We note that this property has been noticed in [CST] where it was employed to show that CA 184 on a ring of size N will come to its invariant state by time not greater than $N/2$.) The set of the invariant measures which are also translation invariant is then easily classified (Remark 2.2). For BA, an extremal measure of this set is supported either by configurations that have only particles with velocity $+1$ or by those that have only particles with velocity -1 , moreover, under such a measure the positions of these particles form a stationary ergodic process on \mathbf{Z} . Applying then Assertions 1.1 and 1.2 to this result one easily characterizes the translation invariant measures that are invariant for CA 184 and SG. For extremal translation invariant measures of CA 184 we calculate the flux of particles through zero. The curve of the flux (as a function of the density of particles $\rho \in [0, 1]$) is $1/2 - |1/2 - \rho|$; we observe that it is known to be $\rho(1 - \rho)$ for TASEP.

The invariant measures of BA which are not translation invariant are called “phase separation” measures. These measures are supported by configurations in which all particles with velocity $+1$ are situated to the left of all those whose velocity is -1 . Our characterization of the phase separation measures is obtained via introducing in BA an extra “second class” particle in the way such that its paths record the positions of particles in BA, and studying the properties of these paths. The second class particle is always between the rightmost particle with velocity $+1$ and the leftmost particle with velocity -1 . It moves together with the latter till it is annihilated by the former. At this moment, the second class particle changes its velocity and starts moving to the right until it meets another particle with velocity -1 . The definition of the second class particle yields the following relations of its path to particle positions in BA: each time interval the second class particle moves to the right (respectively, left), is equal to the distance between two subsequent particles with velocity -1 (respectively, $+1$). As for the relative positions of particles with velocity -1 to those with velocity $+1$, it must be such that the their annihilation times form a point process identical to that formed by the times of change of velocity of the second class particle. These relations are employed in Theorem 2.3 which states that BA is distributed due to a phase separation invariant measure if and only if the second class particle paths belong to the set $E := \{e = (e_i)_{i \in \mathbf{Z}} : e_{2i} \in 2\mathbf{Z} \forall i \in \mathbf{Z}, \text{ and } e_i - e_{i+1} \in \{-1, 1\} \forall i \in \mathbf{Z}\}$ and have the distribution which is invariant with respect to the shift of E by 2 (i.e. $e_i \rightarrow e_{i+2}$). Above e_i is interpreted as the position of the second class particle at time i . For CA 184, a similar results holds with the only difference that now the invariance is with respect to shift by 1, and $E := \{e = (e_i)_{i \in \mathbf{Z}} : e_i - e_{i+1} \in \{-1, 1\} \forall i \in \mathbf{Z}, \text{ and if for some } i \text{ and } j, j > i \text{ it holds that } e_i - e_{i-1} = -(e_{i+1} - e_i) = e_{i+2} - e_{i+1} = \dots = e_j - e_{j-1} = -(e_{j+1} - e_j) \text{ then } j - i \text{ is odd}\}$, other words, the second class particle motion in CA 184 is a time-homogeneous process with the state space E . We note that the ideas similar to those just presented, have been used in [FR] to construct phase separation measures for a so-called Boghosian Levermore cellular automaton.

In Section 3 we review and generalize results from [BF]. Theorem 3.1 presents a hydrodynamic limit for the shape of surface in SG. It says that if in SG $\{f_n(\cdot), n \in \mathbf{N}\}$ the distribution of $c_n f_0(n \cdot)$ converges to some process $W(\cdot)$ for some sequence of numbers $\{c_n\}$ then $c_n f_n^*(n \cdot)$ converges to the process $W^{\min}(\cdot)$ which is defined by $W^{\min}(x) :=$

$\min\{W(y), x-1 \leq y \leq x+1\}, \forall x \in \mathbf{R}$; above f_n^* denotes a particular function whose shape coincides with the shape of f_n . The cornerstone of this result is the following property of SG: if for a function $f(\cdot) \in \mathcal{R}$ we define the function $g(\cdot)$ by $g(x) = \min\{f(y), x-1 \leq y \leq x+1\}$ then $g(\cdot)$ and $\hat{f}(\cdot)$ have the same shape. With help of Assertions 1.1, 1.2 we then derive from Theorem 3.1 the $time \rightarrow \infty$ limits for the particle distribution in BA (Theorem 3.2) and in CA 184 (Theorem 3.3), under appropriate rescaling (i.e. hydrodynamic limits). These theorems express the limit of distributions of particles in BA and CA 184 in terms of a limit of their counting processes. A counting process for BA $\{\zeta_n, n \in \mathbf{N}\}$ is, by definition, the process $\{T_{BA,SG}(\zeta_n), n \in \mathbf{N}\}$. Since the latter closely relates to SG, as specified in Assertion 1.2 and Remark 1.2, then Theorem 3.2 is immediate from Theorem 3.1. To obtain a counterpart for CA 184, just a simple trick is required. The results of Section 3 have two natural applications which we demonstrate via examples. Example 3.1 shows how BA and CA 184 may be employed in order to find the law of $W^{\min}(\cdot)$ for a given process $W(\cdot)$. Example 3.2 demonstrates a simple way to find approximately the particle distribution in BA or CA 184 for large times.

In Section 4 we calculate the rate of convergence of CA 184 starting from the Bernoulli $1/2$ product measure, to its invariant state. This is equivalent to calculating the rate of the decay of particles in BA with a particular initial state. The argument we employ is not new. It has been used to estimate analogous rates in various models (see [Fi], [KS], [EF]). This argument is based on estimating a certain quantity which we specify in Remark 4.2. We observe (in Remark 4.1) that the result discussed in this section could be also obtained using the tools presented in Section 3.

We consider the results of Theorems 2.3, 2.4, 3.3 to be our novel contribution to the study of the dynamics of BA, CA 184 and SG. Theorems 2.1, 2.2 and 3.1, 3.2 have original form but the bulk of their proofs is based on ideas and techniques that have appeared in literature related to the field.

2. INVARIANT MEASURES

In this section, we give the necessary and sufficient condition for a measure to be invariant for BA (Theorem 2.1) and for CA 184 (Theorem 2.2); the corresponding result for the shape in SG may be easily obtained from that for BA (by using T from Assertion 1.2) and thus, will not be presented here. Those invariant measures which are translation invariant are characterized in Remark 2.2, and those which are not translation invariant are characterized in Theorems 2.3 and 2.4. At the end of Section 2 we calculate the flux of particles in CA 184 when it is distributed with an invariant and translation invariant measure.

We start with some definitions. Elements from $\{0,1\}^{\mathbf{Z}}$ and from $\{-1,0,1\}^{\mathbf{Z}}$ will be called *configurations*, and their values at sites of \mathbf{Z} will be interpreted in terms of particles as indicated in Introduction. A-particles of BA will be called simply *particles*, and those whose velocity is $+1$ (resp., -1) will be called *positive* (resp., *negative*). We set $\Theta := \{-1,0,1\}^{\mathbf{Z}}$. By \mathcal{I} we will denote the set of the measures that are invariant for the dynamics of BA. To say $\mu \in \mathcal{I}$ means that $\mu(A^{-1}(U)) = \mu(U)$ for any cylinder set $U \subset \Theta$ (that is, a set of the form $U := \{\zeta \in \Theta : \zeta(i_1) = a_1, \dots, \zeta(i_k) = a_k\}$ for some $k \in \mathbf{N}, i_1, i_2, \dots, i_k \in \mathbf{Z}$

and $a_1, a_2, \dots, a_k \in \{-1, 0, 1\}$.) By τ_1 we will denote the translation of \mathbf{Z} by 1 to the right. By \mathcal{T} we shall denote the set of those measures on $\{-1, 0, 1\}^{\mathbf{Z}}$ that are translation invariant, i.e., $\mu(\tau_1^{-1}U) = \mu(U)$ for any cylinder set $U \subset \Theta$. By ϕ we shall denote the configuration that does not have any particle: $\phi(i) = 0 \forall i \in \mathbf{Z}$. By $\Theta_+ := \{0, 1\}^{\mathbf{Z}} \setminus \{\phi\}$ (resp., $\Theta_- := \{-1, 0\}^{\mathbf{Z}} \setminus \{\phi\}$) we denote the set of those configurations of Θ which have solely positive (respectively, negative) particles. By \mathcal{P}_+ (resp., \mathcal{P}_-) we denote the set of measures concentrated on Θ_+ (resp., on Θ_-). Given a configuration $\zeta \in \Theta$, we will call two particles of ζ *subsequent*, if ζ does not have any other particle between these two. A pair of subsequent particles is said to be *converging* if the leftmost particle of the pair is positive and the rightmost one is negative. The *diverging* type is introduced in analogous way. By Θ_s we denote the set of those configurations $\zeta \in \Theta$ which satisfy the following two conditions

- (i) there is only one pair of subsequent converging particles;
- (ii) both the number of positive particles and the number of negative ones in ζ are infinite.

By \mathcal{S} we will denote the set of measures concentrated on Θ_s . These measures are customarily called *phase separation measures*.

Theorem 2.1. $\mu \in \mathcal{I}$ if and only if for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \geq 0$ such that $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1$ it holds that

$$\mu = \alpha_1 \mu_1 + \alpha_2 \mu_2 + \alpha_3 \mu_3 + \alpha_4 \delta_\phi, \quad (2.1)$$

whereas $\mu_1 \in \mathcal{P}_+ \cap \mathcal{T}$, $\mu_2 \in \mathcal{P}_- \cap \mathcal{T}$, $\mu_3 \in \mathcal{S} \cap \mathcal{I} \cap \mathcal{T}^c$.

Remark 2.1. \mathcal{T}^c means the complement to \mathcal{T} . We wrote $\mathcal{S} \cap \mathcal{I} \cap \mathcal{T}^c$ in order to emphasize that $\mathcal{S} \cap \mathcal{I}$ cannot contain a translation invariant measure. This fact can be established by the following reasoning: Let $pos(\zeta)$ and $neg(\zeta)$ denote the positions of respectively, the positive and the negative particles of the unique pair of converging particles from $\zeta \in \Theta_s$. Define the function $X : \Theta_s \rightarrow \mathbf{Z} \cup (\mathbf{Z} + 1/2)$ by $X(\zeta) := (neg(\zeta) - pos(\zeta))/2$. Now, if ζ is distributed by a translation invariant measure then the distribution of $X(\zeta)$ would be invariant with respect to translations of \mathbf{Z} , which is impossible.

Remark 2.2. The set $(\mathcal{P}_+ \cap \mathcal{T}) \cup \delta_\phi$ (and $(\mathcal{P}_- \cap \mathcal{T}) \cup \delta_\phi$, by analogy) is by its definition, the set of all translation invariant measures on $\{0, 1\}^{\mathbf{Z}}$. The extremal points of this set is known to coincide with the set of all ergodic (with respect to τ_1) measures on $\{0, 1\}^{\mathbf{Z}}$.

Proof of Theorem 2.1. Observe that

$$\text{if } \zeta \in \Theta_+ \text{ then } \tau_1(\zeta) = A(\zeta) \quad (2.2)$$

It follows then easily from (2.2) that if $\mu_1 \in \mathcal{P}_+ \cap \mathcal{T}$ then $\mu_1 \in \mathcal{I}$. Analogously, if $\mu_2 \in \mathcal{P}_- \cap \mathcal{T}$ then $\mu_2 \in \mathcal{I}$. Since μ_3 in (2.1) is picked from a subset of \mathcal{I} and since obviously $\delta_\phi \in \mathcal{I}$ then μ defined by (2.1) must belong to \mathcal{I} . This completes the proof of the “if” part of the theorem. The “only if” part is more difficult. It will be proved with the aid of the following

Assertion 2.1. Let $\zeta_0 \in \Theta$ and $m \in \mathbf{N}$ be arbitrarily fixed. Let ζ_m denote the configuration of particles in the BA at time m , starting from ζ_0 , i.e., $\zeta_m = A^m \zeta_0$. Then the

distance between the particles of any pair of diverging particles from ζ_m is not less than $2m + 1$.

Proof of Assertion 2.1. Find any pair of diverging particles in ζ_m . Observe that the initial position of the negative particle of this pair must be to the left of that of the positive one, because if not then these particle would have annihilated each other by time m . The assertion follows from this observation and from the fact that these particles diverge with velocity 2. ♣

Continuation of the proof of Theorem 2.1. Assertion 2.1 provides that if $\mu \in \mathcal{I}$ then it gives weight zero to any configuration that has at least one pair of diverging particles. Thus such μ must be concentrated on $\Theta_+ \cup \Theta_- \cup \Theta_s \cup \{\phi\}$. Since the dynamics A does not mix these sets (that is, there is no a configuration ζ from one of these sets such that $A\zeta$ belongs to another one) then the expansion (2.1) follows. ♣

Let T be defined by (1.3). We define $\Lambda \subset \Theta$ as the image of $\{0, 1\}^{\mathbb{Z}}$ by T . Namely, $\Lambda := \{\zeta \in \Theta : \exists \eta \in \{0, 1\}^{\mathbb{Z}} \text{ such that } T\eta = \zeta\}$. From this definition and that of T , Λ is easily characterized: $\Lambda := \{\zeta \in \Theta : \text{the distance between any subsequent particles with the same velocity is odd, and the distance between any two subsequent particles with opposite velocities is even}\}$. The set Λ is used in the following statement:

Theorem 2.2. *A measure μ on $\{0, 1\}^{\mathbb{Z}}$ is invariant for CA 184 if and only if*

$$\mu = \alpha_1 \mu_1 + \alpha_2 \mu_2 + \alpha_3 \mu_3 + \alpha_4 \left(\frac{\delta_o + \delta_e}{2} \right), \text{ for some } \alpha_1, \alpha_2, \alpha_3, \alpha_4 \geq 0, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1$$

where the measures μ_1 and μ_2 are translation invariant and are concentrated on the sets $T^{-1}(\Theta_+ \cap \Lambda)$ and $T^{-1}(\Theta_- \cap \Lambda)$ respectively; the measure μ_3 is invariant for CA 184 but not translation invariant and is concentrated on $T^{-1}(\Theta_s \cap \Lambda)$; the measure δ_o (resp., δ_e) gives mass 1 to the configuration o (resp., e) where o and $e \in \{0, 1\}^{\mathbb{Z}}$ are such that $o(i) = 1$, $e(i) = 0$, when i is odd, and $o(i) = 0$, $e(i) = 1$ when i is even, $i \in \mathbb{Z}$.

The proof of Theorem 2.2 comes out by a reformulation of the proof of Theorem 2.1 in terms of CA 184. This reformulation is based on the property of T stated in Assertion 1.1, and the facts that $T(\{0, 1\}^{\mathbb{Z}}) = \Lambda$ and that T is invertible on $\{0, 1\}^{\mathbb{Z}} \setminus \{o, e\}$. The only care thus, should be taken because of the fact that $T(o) = T(e) = \phi$. But this causes no difficulties in the mentioned reformulation because we have that $C(o) = e$ and $C(e) = o$ and as a consequence of this, we also have that if μ is invariant for CA 184 then $\mu(o) = \mu(e)$. ♣

We now characterize the set of invariant measures of BA which are not translation invariant. Set

$$E := \{e = (e_i)_{i \in \mathbb{Z} \cup \mathbb{Z} + \frac{1}{2}} : e_i \in \mathbb{Z} \forall i \in \mathbb{Z} \text{ and } e_i - e_{i+\frac{1}{2}} \in \{-\frac{1}{2}, \frac{1}{2}\} \forall i \in \mathbb{Z} \cup \mathbb{Z} + \frac{1}{2}\}$$

and define the shift $\tau_E : E \rightarrow E$ by $\tau_E(e) = e' \Leftrightarrow e'_i = e_{i+1} \forall i$. Denote then by \mathcal{E}_τ the set of the measures on E which are invariant with respect to the shift τ_E .

Theorem 2.3. *There is a bijection $F^* : \mathcal{S} \cap \mathcal{I} \rightarrow \mathcal{E}_\tau$. The mapping F^* will be constructed explicitly in the proof of the theorem.*

Proof. The idea is to introduce a second class particle in BA and to show that BA is distributed by an invariant phase separation measure if and only if the second class particle's motion has invariant distribution with respect to shifts of time. The second class particle in BA is defined by the following rules: (i) it is an extra particle that does not affect the evolution of other particles, and (ii) it moves with the velocity that changes from $+1$ to -1 and back depending on the BA particles that the second class particle meets, namely: (ii-a) when the second class particle meets a negative particle it changes its velocity to -1 and keeps going together with the met particle; (ii-b) at the time this met particle is annihilated, the second class particle changes its velocity to $+1$. One more postulate is: (iii) if two particles annihilate at a time $n \in \mathbf{Z} \cup \mathbf{Z} + \frac{1}{2}$ then at this time, the second class particle is at the annihilation point and changes its velocity from -1 to $+1$. We want the rules (i)-(iii) to define correctly and uniquely the second class particle and moreover, we want each trajectory of the second class particle to determine uniquely the configuration of BA at any time. To achieve these objectives we have to consider BA with double-infinite time and to record the second class particle position at times from the set $\mathbf{Z} \cup \mathbf{Z} + \frac{1}{2}$. Note that $\mathbf{Z} + \frac{1}{2}$ has to be added because annihilations may occur at times from this set. Also note that the distribution of the second class particle trajectories turns out to be invariant with respect to the shift of time by 1 but not necessarily to the shift of time by $\frac{1}{2}$. The details are presented below.

We start introducing a terminology which we shall need to construct BA with double infinite time. For an element ω from $\{-1, 0, 1\}^{\mathbf{Z} \times \mathbf{Z}}$, let us write $\omega_n(m)$ for its values at $(n, m) \in \mathbf{Z} \times \mathbf{Z}$. Let us also write ω_n for the restriction of $\omega \in \{-1, 0, 1\}^{\mathbf{Z} \times \mathbf{Z}}$ obtained by fixing its first coordinate to n ; observe $\omega_n \in \{-1, 0, 1\}^{\mathbf{Z}}$. Define then $\Omega := \{\omega \in \{-1, 0, 1\}^{\mathbf{Z} \times \mathbf{Z}} : \omega_n \in \Theta_s \text{ and } \omega_{n+1} = A\omega_n \forall n\}$. Fix now an arbitrary $\mu \in \mathcal{S} \cap \mathcal{I}$. There is a standard procedure to construct a probability measure \mathbf{P}_Ω^μ (on an appropriate σ -algebra \mathcal{F} of subsets of Ω) with the following properties (2.3), (2.4):

$$\mathbf{P}_\Omega^\mu[\omega : \omega_n \in D] = \mu[D], \quad \forall n \in \mathbf{Z} \text{ and } \forall \text{ cylinder } D \subset \{-1, 0, 1\}^{\mathbf{Z}} \quad (2.3)$$

If we define the shift operator $\tau_\Omega : \Omega \rightarrow \Omega$ acting by $(\tau_\Omega(\omega))_n(m) := \omega_{n-1}(m), \forall n, m$, then

$$\mathbf{P}_\Omega^\mu[D] = \mathbf{P}_\Omega^\mu[\tau_\Omega^{-1}(D)], \quad \forall \text{ cylinder } D \subset \{-1, 0, 1\}^{\mathbf{Z} \times \mathbf{Z}} \quad (2.4)$$

\mathbf{P}_Ω^μ will be called *BA with double infinite time and the marginal distributions μ* . Such a choice of the name becomes obvious if one interprets ω as a trajectory of a certain process, and $\omega_n(m)$ as its value at time n and the location m . We recall that it is known that given $\mu \in \mathcal{S} \cap \mathcal{I}$, there is a unique \mathbf{P}_Ω^μ satisfying (2.3), (2.4), and, moreover, given any probability measure \mathbf{P} on Ω which is invariant with respect to τ_Ω , there is a unique $\mu \in \mathcal{S} \cap \mathcal{I}$ such that $\mathbf{P}_\Omega^\mu = \mathbf{P}$, other words, $\mu \rightarrow \mathbf{P}_\Omega^\mu$ is a bijection from $\mathcal{S} \cap \mathcal{I}$ to the set of all probability measures on Ω which are invariant in respect to τ_Ω .

By certain technical reasons that will become clear soon, we need to have a record of the positions of particles in BA also at times from the set $\mathbf{Z} + \frac{1}{2}$. We thus associate to

each $\omega \in \Omega$ an element $\omega' \in \{-1, 0, 1\}^{\mathbf{Z} \cup \mathbf{Z} + \frac{1}{2} \times \mathbf{Z} \cup \mathbf{Z} + \frac{1}{2}}$ by the following rule: for each $n \in \mathbf{Z}$ we set

$$\omega'_n(m) = \begin{cases} \omega_n(m), & \text{when } m \in \mathbf{Z} \\ 0, & \text{when } m \in \mathbf{Z} + \frac{1}{2} \end{cases} \quad (2.5)$$

and

$$\omega'_{n+\frac{1}{2}}(m) = \begin{cases} 0, & \text{if } m \in \mathbf{Z} \\ 1, & \text{when } m \in \mathbf{Z} + \frac{1}{2} \text{ and if } \omega_n(m - \frac{1}{2}) = 1 \text{ and } \omega_n(m + \frac{1}{2}) \neq -1 \\ -1, & \text{when } m \in \mathbf{Z} + \frac{1}{2} \text{ and if } \omega_n(m + \frac{1}{2}) = -1 \text{ and } \omega_n(m - \frac{1}{2}) \neq 1 \\ 0, & \text{otherwise} \end{cases} \quad (2.6)$$

The relation between ω and ω' may be explained in the following manner: Suppose that two persons observe the evolution of the same process BA. Suppose that one of them records the states of the process at times \mathbf{Z} while the other one records the states of the process at times $\mathbf{Z} \cup \mathbf{Z} + \frac{1}{2}$. Then, ω and ω' are the histories recorded by respectively, the first and the second person. Having this interpretation in mind and observing that a configuration of particles in BA at time $n \in \mathbf{Z} \cup \mathbf{Z} + \frac{1}{2}$ determines uniquely the configuration at time $n + \frac{1}{2}$ we conclude that the mapping $\omega \rightarrow \omega'$ is a bijection.

We are now in a position to define the mapping $F : \Omega \rightarrow E$. To obtain $e = F(\omega)$, we construct first ω' by the rules (2.5)-(2.6) and then, for each $n \in \mathbf{Z} \cup \mathbf{Z} + \frac{1}{2}$, we define e_n by the following rules:

- 1 if a point $m \in \mathbf{Z} \cup \mathbf{Z} + \frac{1}{2}$ is an annihilation place of two particles at time n in ω' (that is, if $\omega'_n(m) = 0, \omega'_{n-\frac{1}{2}}(m - \frac{1}{2}) = 1, \omega'_{n-\frac{1}{2}}(m + \frac{1}{2}) = -1$) then we set $e_n = m$;
- 2 if $e_{n-\frac{1}{2}} = m$ and $\omega'_{n-\frac{1}{2}}(m) = -1$ then we set $e_n = e_{n-\frac{1}{2}} - \frac{1}{2}$;
- 3 if $e_{n-\frac{1}{2}} = m$ and $\omega'_{n-\frac{1}{2}}(m) = 0$ then we set $e_n = e_{n-\frac{1}{2}} + \frac{1}{2}$.

An important observation is that F is a bijection. This follows from two facts. The first one is that $\omega \rightarrow \omega'$ is a bijection, as mentioned above. The second fact is that ω' determines uniquely $e \in E$ and vice versa. This fact may be verified straightforward. We observe that in order that $F^{-1}e$ be uniquely determined, it was essential that the elements of E be indexed by the set $\mathbf{Z} \cup \mathbf{Z} + \frac{1}{2}$. (We suggest a reader to choose an arbitrary $\epsilon \in E$, to construct then $\varepsilon \in E$ by changing the value of ϵ at $\frac{1}{2}$, and to verify then that $F^{-1}(\epsilon) \neq F^{-1}(\varepsilon)$.) The mapping F introduces naturally a measure \mathbf{P}_E^μ on E via $\mathbf{P}_E^\mu[D] := \mathbf{P}_\Omega^\mu[F^{-1}D], \forall$ cylinder $D \subset E$. But because of the property

$$\tau_\Omega(F^{-1}(e)) = F^{-1}(\tau_E(e))$$

that follows from our constructions and because of (2.4), we conclude that $\mathbf{P}_E^\mu[D] = \mathbf{P}_E^\mu[\tau_E^{-1}D], \forall D \in E$, that is \mathbf{P}_E^μ is from \mathcal{E}_τ . We thus define F^* by $F^*(\mu) := \mathbf{P}_E^\mu$, for $\mu \in \mathcal{S} \cap \mathcal{I}$. The fact that F^* is a bijection follows from that the mappings F and $\mu \rightarrow \mathbf{P}_\Omega^\mu$ are bijections. ♣

Recall Theorem 2.2. Let us give the name *invariant phase separation measure of CA 184* to an invariant measure of CA 184 which is concentrated on $T^{-1}(\Theta_s \cap \Lambda)$. The following result characterizes these measures.

Theorem 2.4. *There is a one-to-one mapping between the set of all invariant phase separation measures of CA 184 and the set of all stationary processes $(X_n)_{n \in \mathbb{Z}}$ with values in \mathbb{Z} which satisfy the conditions*

- (a) $X_{n+1} - X_n \in \{-1, 1\}$, for all n ;
- (b) if for some i, j , $j > i$, it holds that $X_i - X_{i-1} = -(X_{i+1} - X_i) = X_{i+2} - X_{i+1} = \dots = X_j - X_{j-1} = -(X_{j+1} - X_j)$ then $j - i$ is odd.

This mapping can be obtained as an appropriate restriction of F^ in the manner outlined in the proof.*

Proof. The mapping T from (1.3) induces a natural mapping between the distributions of CA 184 and that of BA. The property that $T : \{0, 1\}^{\mathbb{Z}} \rightarrow \Lambda$ is a bijection (recall, Λ has been defined before the statement of Theorem 2.2) then yields that the induced mapping is a bijection between the set of all invariant phase separation measures of CA 184 and a subset of $(\mathcal{S} \cap \mathcal{I})$ which we shall denote by $(\mathcal{S} \cap \mathcal{I})_{184}$.

Thus, Theorem 2.4 is proved, whence we show that $F^*((\mathcal{S} \cap \mathcal{I})_{184})$ is the set of the processes X as defined in the theorem's statement. This can be done directly via repeating the proof of Theorem 2.3, but applying it to $(\mathcal{S} \cap \mathcal{I})_{184}$ instead of $\mathcal{S} \cap \mathcal{I}$. For this application to be correct, it is necessary to note the difference between these two sets. It follows from the property of T mentioned in the above paragraph that a measure from $\mathcal{S} \cap \mathcal{I}$ is in $(\mathcal{S} \cap \mathcal{I})_{184}$ if and only if it gives weight 0 to the set $\Lambda^c = \{\zeta \in \{-1, 0, 1\}^{\mathbb{Z}} : \text{the distance between any subsequent particles with the same velocity is even or the distance between any two subsequent particles with opposite velocities is odd}\}$. This suggests the following two modifications in construction of $F^*((\mathcal{S} \cap \mathcal{I})_{184})$ compared to that of $F^*(\mathcal{S} \cap \mathcal{I})$: First, we do not need to record the second class particle positions at times $\mathbb{Z} + \frac{1}{2}$, because there are no annihilations at these times. Second, the set of paths of the second class particle should be modified so that when a path from this set is translated into corresponding configuration of annihilation particles, then the latter must belong to Λ^c . To satisfy these requirements, we imposed the conditions (a) and (b) on the process X . The rest of the proof goes almost identically to that of Theorem 2.3 and thus, will be omitted. ♣

Remark 2.3. We recall (from [L] and [Y], respectively) that in the case of TASEP and CA&N, the set of phase separation measures is $\{\nu^{(n)}, -\infty < n < \infty\}$, where $\nu^{(n)}$ gives mass 1 to the configuration $\eta^{(n)}$ such that $\eta^{(n)}(x) = 1 \forall x \geq n$ and $\eta^{(n)}(x) = 0 \forall x < n$.

Flux of particles in CA 184 which is distributed due to an translation invariant measure μ which is invariant for CA 184, is defined as

$$\mu\{\eta_n(1)(1 - \eta_{n-1}(1))\}$$

We shall now show that the flux of particles in CA 184 is given by $1/2 - |1/2 - \rho|$, if CA 184 distributed by such a measure μ , where ρ denotes the density of particles. (Here the density of particles is understood as $\mu\{\eta_n(i)\}$ which certainly does not depend on n and i .) Let μ be such a measure and consider first the case $\rho < 1/2$. Then Theorem 2.2 implies that $\mu \in T^{-1}(\Theta_- \cap \Lambda)$ which in turn, implies that $\mu\{\eta_n(i) = 0 \mid \eta_n(i-1) = 1\} = 1, \forall i \forall n$. Thus the flux of particles is ρ . In the case when $\rho > 1/2$, the flux of particles is $1 - \rho$. This results from the following reasoning: We define the flux of holes as $\mu\{(1 - \eta_n(0))\eta_{n-1}(0)\}$

and derive that it equals $1 - \rho$, this result follows by adapting the reasoning that have been used for the case $\rho < 1/2$. We then observe that the flux of holes must be equal to the flux of particles (this observation may be justified in one of the following ways: either using the fact that μ is invariant, or from a direct comparison of the definitions of the fluxes). It is left to consider the case $\rho = 1/2$ which occurs as follows from Theorem 2.2, when $\mu = 1/2(\delta_o + \delta_e)$. When it happens, then flux is obviously $1/2$.

3. HYDRODYNAMIC LIMIT.

This section concerns the limit (as *time* $\rightarrow \infty$) shape of surface in SG (Theorem 3.1) and the limit distribution of particles in BA (Theorem 3.2) and in CA 184 (Theorem 3.3). These results are called hydrodynamic limits because along with *time* tending to ∞ we apply a certain rescaling that depends on *time*, to the quantities whose temporal limit is of interest. We also demonstrate here applications of these results (Examples 3.1, 3.2).

For $y \geq 0$, let us denote by M_y the operator that brings a function $f(\cdot) : \mathbf{R} \rightarrow \mathbf{R}$ to the function $g(\cdot) : \mathbf{R} \rightarrow \mathbf{R}$ in the way such that

$$g(x) = (M_y f)(x) = \min\{f(z) : x - y \leq z \leq x + y\}, \quad \forall x \in \mathbf{R}$$

Let us then introduce *Surface Modification Process* (SM) as the name of a sequence $\{f_n(\cdot), n \in \mathbf{N}\}$ of random functions from \mathcal{R} such that $f_{n+1}(\cdot) = (M_1 f_n)(\cdot)$ for each $n \in \mathbf{N}$. One then can check straightforwardly the following property:

$$\text{for any } f \in \mathcal{R} \text{ there is an } \alpha \in \mathbf{R} \text{ such that } (M_1 f)(\cdot) = \hat{f}(\cdot) + \alpha \quad (3.1)$$

whereas the construction rule of $\hat{f}(\cdot)$ has been specified in the part of Introduction that defined the Surface growth process. The above (3.1) says that at each time, the shape of the states of SG and SM is the same, provided they started from the same state. However, SM is more handy for formulating the hydrodynamic limit for this shape. To state this result, we need first to introduce some notations: (1) For a process $W(x)$, $x \in \mathbf{R}$, we will denote by $W^{\min}(\cdot)$ the process defined by $W^{\min}(x) := (M_1 W)(x)$, $\forall x \in \mathbf{R}$. (2) The convergence of processes discussed in this section and throughout the paper, is understood as the weak convergence of their distributions on $\mathcal{C}[a, b]$, the set of continuous functions on $[a, b]$, for any $-\infty < a < b < \infty$ (see [Bi]).

Theorem 3.1 (hydrodynamic limit for the shape of surface in SG). *Let $f_0(\cdot)$ be a random function with the state space \mathcal{R} such that $c_n f_0(n \cdot) \rightarrow W(\cdot)$, as $n \rightarrow \infty$, for some sequence $\{c_n, n \in \mathbf{N}\}$ of real numbers and some process $W(x)$, $x \in \mathbf{R}$. Then the SM $\{f_n(\cdot), n \in \mathbf{N}\}$ satisfies the following property*

$$c_n f_n(n \cdot) \rightarrow W^{\min}(\cdot) \quad (3.2)$$

Proof. First, we note that for any function $h : \mathbf{R} \rightarrow \mathbf{R}$, the modulus of continuity (which definition one finds in [Bi]) of $M_1 h$ does not exceed that of h ; this property of M_1 can

be verified straightforward. This fact allows us to apply Theorem 5.1 from [Bi] to the assumption $c_n f_0(n \cdot) \rightarrow W(\cdot)$ to conclude that

$$M_1(c_n f_0(n \cdot))(\cdot) \rightarrow (M_1 W)(\cdot) = W^{\min}(\cdot) \quad (3.3)$$

We must comment on the usage of two "·" in (3.3) in order to avoid a possible confusion in its interpretation: $M_1(c_n f_0(n \cdot))(\cdot)$ means the function obtained by rescaling f_0 by c_n along the axis of ordinates and by n^{-1} along the axis of abscissas, and then, by applying M_1 . It may be verified directly that for any continuous function h and any constant c ,

$$(M_1 c h(n \cdot))(x) = c(M_n h)(nx), \quad \forall x \in \mathbf{R},$$

and also that n subsequent applications of M_1 is equivalent to one application of M_n . Thus, (3.2) follows from (3.3). ♣

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be an abstract probability space. We shall say that a random function $f_n(\cdot) : \Omega \rightarrow \mathcal{R}$ is a *counting profile* of $\zeta_n : \Omega \rightarrow \{-1, 0, 1\}^{\mathbf{Z}}$ (resp., $\eta_n : \Omega \rightarrow \{0, 1\}^{\mathbf{Z}}$) if $f(k)[\omega] - f(k-1)[\omega] = \zeta(k)[\omega]$ (resp., $f(k)[\omega] - f(k-1)[\omega] = -\eta(k)[\omega]$), $\forall k \in \mathbf{Z}$ (certainly, it is enough to specify the values of a function from \mathcal{R} just on \mathbf{Z}) and $\forall \omega \in \Omega$, where we denoted by $Y[\omega]$ the realization on $\omega \in \Omega$ of a function Y defined on Ω . We say that $\{f_n(\cdot), n \in \mathbf{N}\}$ is a *counting process* of BA $\{\zeta_n, n \in \mathbf{N}\}$ (resp., CA 184 $\{\eta_n, n \in \mathbf{N}\}$) if for each n , $f_n(\cdot)$ is a counting profile of ζ_n (resp., η_n).

From the definitions just introduced, Assertion 1.2 and (3.1), we conclude that if a random function $f_0(\cdot)$ is a counting profile of a random configuration ζ_0 then SM $\{f_n(\cdot), n \in \mathbf{N}\}$ is a counting process for BA $\{\zeta_n, n \in \mathbf{N}\}$. Thus, Theorem 3.1 leads to the following

Theorem 3.2 (hydrodynamic limit for particle distribution in BA). *Let $\{\zeta_n, n \in \mathbf{N}\}$ be a BA. Assume there is $f_0(\cdot)$, a counting profile of ζ_0 , such that $c_n f_0(n \cdot) \rightarrow W(\cdot)$ as $n \rightarrow \infty$, for some sequence $\{c_n, n \in \mathbf{N}\}$ of real numbers and some process $W(x), x \in \mathbf{R}$. Then there is a counting process $\{f_n(\cdot), n \in \mathbf{N}\}$ of BA such that $c_n f_n(n \cdot) \rightarrow W^{\min}(\cdot)$. Precisely to say, this counting process is SM with the initial state $f_0(\cdot)$.*

The next result is a counterpart of Theorem 3.2 for CA 184. Note that it contains an additional assumption " $c_n \rightarrow 0$ as $n \rightarrow \infty$ ". In respect to this note, we observe that in all applications of hydrodynamic type limit theorems we are aware of, this condition is satisfied.

Theorem 3.3 (hydrodynamic limit for particle distribution in CA 184). *Let $\{\eta_n, n \in \mathbf{N}\}$ be a CA 184. Assume there is $g_0(\cdot)$, a counting profile of η_0 , such that $c_n g_0(n \cdot) \rightarrow W(\cdot)$ as $n \rightarrow \infty$, for some process $W(x), x \in \mathbf{R}$, and some sequence $\{c_n, n \in \mathbf{N}\}$ of real numbers satisfying $c_n \rightarrow 0$ as $n \rightarrow \infty$. Then there is a counting process $\{g_n(\cdot), n \in \mathbf{N}\}$ of CA 184 such that $c_n g_n(n \cdot) \rightarrow W^{\min}(\cdot)$. The construction rule of $g_n(\cdot)$ will be specified in the proof of this theorem.*

Proof. It is straightforward to see that

$$\begin{aligned} & \text{if } T\eta = \zeta \text{ for some } \eta \in \{0, 1\}^{\mathbf{Z}} \text{ and } \zeta \in \{-1, 0, 1\}^{\mathbf{Z}}, \text{ where } T \text{ is from (1.3),} \\ & \text{and if } g \text{ and } f \text{ are counting profiles of } \eta \text{ and } \zeta \text{ respectively,} \\ & \text{satisfying } g(0) = f(0) \text{ then } |f(x) - g(x)| \leq 1 \forall x \in \mathbf{R} \end{aligned} \quad (3.4)$$

Let $\zeta_0 := T\eta_0$, where T is from (1.3), and let $f_0(\cdot)$ be the counting profile of ζ_0 such that $f_0(0)[\omega] = g_0(0)[\omega]$, $\forall \omega$. By theorem's assumptions and (3.4), $c_n f_0(n\cdot) \rightarrow W(\cdot)$. The latter and Theorem 3.2 yield that $c_n f_n(n\cdot) \rightarrow W^{\min}(\cdot)$ for an appropriate counting process $\{f_n(\cdot), n \in \mathbf{N}\}$ of the BA $\{\zeta_n, n \in \mathbf{N}\}$. This conclusion, (3.4) and the assumption $c_n \rightarrow 0$ then yield that $c_n g_n(n\cdot) \rightarrow W^{\min}(\cdot)$, if we define $g_n(\cdot)$ to be the counting profile of η_n such that $g_n(0)[\omega] = f_n(0)[\omega]$, $\forall \omega$. \clubsuit

Two applications of the above results are quite natural. We shall demonstrate them in examples below.

Example 3.1 demonstrates that given a process $W(\cdot)$, the distribution of $W^{\min}(\cdot)$ can be found by considering an appropriate cellular automaton. This example is a brief review of a result contained in [BF] in which $W(\cdot)$ was taken to be $B(\cdot)$, the one-dimensional Brownian motion with $B(0) = 0$.

Let $\zeta_n, n \in \mathbf{N}$, be the BA whose initial state has the following distribution:

$$\zeta_0(i), i \in \mathbf{Z}, \text{ are i.i.d. with } \mathbf{P}[\zeta_0(i) = 1] = \mathbf{P}[\zeta_0(i) = -1] = 1/2 \quad (3.6)$$

It is then not hard to find exactly the distribution of $\zeta_n = A^n \zeta_0$. It is as follows: it is invariant with respect to translations and reflections of \mathbf{Z} , and given a site i is occupied by a particle, the probability that the nearest particle to its right has the same (opposite) velocity is $(1 + u_{2n})^{-1}$ (resp., $u_{2n}(1 + u_{2n})^{-1}$) and can be expressed exactly as a function of u_{2n} , where u_{2n} is the probability that a simple symmetric one dimensional random walk returns to the origin at time n ; this expression will depend on the velocities of the particles under the consideration. For details we refer a reader to Theorem 1 in [BF]. Two facts contributed to the success in finding the law of ζ_n in such an exact form. First, a simple expression for the distance between a given particle and its annihilating companion (this is the essence of Assertion 4.2 presented in Section 4). Second, the relation of the distribution of this distance to u_{2n} which stems from a particular choice of the law of ζ_0 , (3.6).

Let now $f_n(\cdot)$ denote the counting profile of ζ_n that satisfies $f_n(0) = 0$. Clearly, the law of $f_n(\cdot)$ can be derived from that of ζ_n presented above. It turns out that the known asymptotics for u_{2n} (see [F]) is sufficiently precise to enable one to derive then the distribution of the process to which $n^{-1/2} f_n(n\cdot)$ converges (for details, see Theorem 2 in [BF]). If f_n were $M_n f_0$ then this law would be of $B^{\min}(\cdot)$, by Theorem 3.2 and because $n^{-1/2} f_0(n\cdot) \rightarrow B(\cdot)$. By technical reasons however (recall Remark 1.2) it is not. Nevertheless, this limit law allows to expose the sample paths of $B^{\min}(\cdot)$. They are described below following [BF].

A generic trajectory of the process $B^{\min}(\cdot)$ is a continuous function and it is composed of decreasing, increasing and flat portions which follow each other in the following cyclic order: flat-increasing-flat-decreasing. A flat portion preceded by a decreasing portion and followed by an increasing one has length 1; such a portion is called *valley*. A flat portion preceded by an increasing portion and followed by a decreasing portion, has the length $\hat{\theta}$ picked from the distribution $\mathbf{P}[\hat{\theta} \leq x] = 2x^{1/2}(1+x)^{-1}$, $x \in [0, 1]$. Such portion is called a *plateau*. An increasing portion is constructed from a realization of an increasing process $\mathcal{L}(x)$, $x \geq 0$, which is stopped at the height that has an exponential-1 law. The process $\mathcal{L}(\cdot)$

is a subordinator whose Lévy measure μ (see [KaS], Chapter 6) has the following form:

$$\mu(d\ell) = \frac{d\ell}{2(\pi\ell^3)^{1/2}}, \quad \ell \in (0, 1), \quad d\ell \subset [0, 1]$$

A decreasing portion being taken with the sign “-”, is a probabilistic replica of an increasing portion. All the portions that compose $B^{\min}(\cdot)$ are independent which means that the random variables involved in the construction of these portions should be taken to be independent.

We observe that what is lacking to have a complete description of $B^{\min}(\cdot)$ is the dependence between $B^{\min}(0)$, the value of this process at $x = 0$, and the form of $B^{\min}(\cdot)$ as a whole. This happened because ζ_n preserves only the form of $(M_n f_0)(\cdot)$. We may present results in this respect in the future ([B]).

Example 3.2. We shall use here our knowledge of the law of $B^{\min}(\cdot)$ in order to find approximately the distribution at time n of CA 184 with a particular initial distribution.

Let $\eta_n, n \in \mathbf{N}$ be a CA 184 whose initial state has the Bernoulli 1/2 distribution, that is

$$\eta_0(i), i \in \mathbf{Z}, \text{ are i.i.d. with } \mathbf{P}[\eta_0(i) = 1] = \mathbf{P}[\eta_0(i) = 0] = 1/2 \quad (3.7)$$

Let next $g_0(\cdot)$ denote the counting profile of η_0 such that $g_0(0) \equiv 0$. Then as it is well known, $n^{-1/2}g_0(n\cdot) \rightarrow B(\cdot)$, where $B(\cdot)$ is as specified in Example 3.1 above. Thus, by Theorem 3.3,

$$n^{-1/2}g_n(n\cdot) \rightarrow B^{\min}(\cdot) \quad (3.8)$$

for an appropriate sequence of counting profiles.

Our programme is now as follows: We shall base on (3.8) to assume that

$$g_n(\cdot) = n^{1/2}B^{\min}(n^{-1}\cdot) \text{ in distribution} \quad (3.9)$$

and shall combine this assumption with the law of $B^{\min}(\cdot)$ presented in Example 3.1 above, in order to derive the picture of the particle distribution in η_n for large n .

Although $g_n(\cdot)$ is never a constant, $n^{-1/2}g_n(\cdot)$ looks like a constant on an interval $I \subset \mathbf{R}$ in which either each odd site of η_n is occupied by a particle while each even site is empty, or vice versa. Let us say that in this case we see a duce pattern of η_n in I . Similar considerations suggest us to call $g_n(\cdot)$ increasing (resp., decreasing) on an interval $I \subset \mathbf{R}$ if η_n does not have a pair of contiguous sites of I occupied by particles (resp., holes) and at both endpoints one finds in η_n a pair of contiguous sites occupied by holes (resp., particles) (the term “hole” used here, is a natural nickname for site free of particle). Let us say that in this case we see a hole dominated (resp., particle dominated) pattern of η_n in I .

Being equipped with the terminology of the above paragraph, we can now present the result of the programme we have layed down. It is the following picture of the distribution of η_n : One will see the space \mathbf{R} as divided in intervals in the way such that each one of them is either a particle dominated, or a hole dominated, or a duce pattern of η_n . The interval endpoints are distributed as homogeneous point process on \mathbf{R} (whose parameters can be calculated explicitly from the law of $B^{\min}(\cdot)$). The intervals follow each other in the

cyclic order: particle dominated – duce – hole dominated – duce – particle dominated. The mean extension of an interval is $Cn + o(n)$, where $C = C_1$ for both a particle dominated and a hole dominated regions, $C = C_2$ for a duce region following a particle dominated region, and $C = C_3$ for a duce region preceding a particle dominated region. The mean number of the pairs of contiguous sites occupied by particles (resp., free of particles) in a particle dominated (resp., hole dominated) region is $C'\sqrt{n} + o(\sqrt{n})$. The numerical values of C_1, C_2, C_3, C' may be calculated from the law of $B^{\min}(\cdot)$.

Going back to the observation at the end of Example 3.1, we draw the reader's attention to the fact that the information on the law of $B^{\min}(\cdot)$ that is absent did not affect the derivation of the law of η_n in the present example. The reason is in that the height of the counting profile $g_n(\cdot)$ (i.e. $g_n(0)$) is indifferent to the distribution of the corresponding η_n .

The picture presented above is not the exact law of η_n because $\sqrt{n}B^{\min}(n^{-1}\cdot)$ might diverge from $g_n(\cdot)$ by at most $o(n^{1/2})$ along the y -axis and by at most $o(n)$ along the x -axis (in fact, they must diverge since the former does not attain its values in \mathcal{R} , and the latter does), and thus, (3.9) is an approximate relation. In fact, our argument does not control the particle amount with the precision higher than $o(n^{1/2})$, that is, the real amount of particles in a region I might differ from that of the picture presented, by at most $|I| \times o(n^{1/2})$. Also, we were unable to give the particle distributions within each pattern, but instead, gave its characteristics in terms of its mean values. These are certainly, disadvantages of the presented approach when compared to a direct calculation of the law of η_n . An advantage is the simplicity: the presented approach requires only to know the law of $W^{\min}(\cdot)$ while a direct calculation requires a lot of work and produces usually cumbersome formulae. We note here that if one wishes to find η_n in the exact form then this could be done in the following way: translate CA 184 to BA (via T from Assertion 1.1), use then the tools developed in [BF] to find the law of BA at time n , and, finally, translate this law back into terms of CA 184.

4. RATE OF CONVERGENCE TO THE EQUILIBRIUM

In this section, we present an argument that estimates both the rate of convergence of CA 184 to its invariant state and the rate of the decay of particle density in BA (the former is the content of Theorem 4.1, and the latter is an auxiliary result established in the course of the proof of this theorem). The way this argument works is demonstrated here on a particular example which is the CA 184 starting from the Bernoulli $1/2$ product measure. Remark 4.2 marks the place in this argument where main technical problems arise, when it is adapted to other initial distributions.

Remark 4.1. Observe that that distribution at time n of the CA 184 studied here, has been approximately found in Example 3.2. From this approximation, one can conclude that the probability to see two fixed contiguous sites of \mathbf{Z} at the same state in CA 184 at time n , decays at least as $\text{const} \times n^{-1/2}$. A refinement of the arguments from Section 3 would reveal that $\text{const} \times n^{-1/2}$ is in fact, the correct asymptotics for this decay. This is exactly the result of Theorem 4.1 presented below. We remark here that these two approaches have much in common, although apparently they are different. Indeed, if one possesses a tool to estimate (4.6) (which is the cornerstone of the proof of Theorem 4.1)

then the same tool may be employed to find the distribution of the process $W^{\min}(\cdot)$ which is the basic ingredient for the approximation argument exhibited in Section 3.

Theorem 4.1. *Consider CA 184 $\{\eta_n, n \in \mathbf{N}\}$ in which η_0 is distributed by the Bernoulli $1/2$ product measure, i.e. $\{\eta_0(i), i \in \mathbf{Z}\}$ are i.i.d. random variables with $\mathbf{P}[\eta_0(i) = 1] = \mathbf{P}[\eta_0(i) = 0] = 1/2$. Let μ_n denote the distribution of η_n . Then μ_n converges as $n \rightarrow \infty$, to $1/2(\delta_o + \delta_e)$ at the rate $n^{-1/2}$. In exact terms this means that for each $k \in \mathbf{N}$ there exist $0 < c'_k < c_k < \infty$ such that*

$$c'_k n^{-1/2} \leq \sup_{U \in \{0,1\}^{\{0,1,\dots,k\}}} |\mu_n(U) - 1/2(\delta_o + \delta_e)(U)| \leq c_k n^{-1/2} \quad (4.2)$$

Proof. Fix $k \in \mathbf{N}$ and define $K := \{0, 1, \dots, k\}$. Define

$$E := \{\eta \in \{0, 1\}^K : \exists i \in \{0, 1, \dots, k-1\} \text{ for which } \eta(i) = \eta(i+1)\}$$

Let $\alpha, \beta \in \{0, 1\}^K$ be the restrictions of respectively, o and e (the configuration e and o have been defined in Theorem 2.2) to K : $\alpha(0) := 0, \beta(0) := 1, \alpha(i+1) := 1 - \alpha(i), \beta(i+1) := 1 - \beta(i), i = 0, 1, \dots, k-1$. Then

$$E \cup \{\alpha, \beta\} = \{0, 1\}^K \quad (4.3)$$

We now observe that since μ_0 is translation invariant then μ_n must be such as well, and this fact leads to

$$\mu_n(\alpha) = \mu_n(\beta) \quad (4.4)$$

Since $1/2(\delta_o + \delta_e)(E) = 0$ and $1/2(\delta_o + \delta_e)(\alpha) = 1/2(\delta_o + \delta_e)(\beta) = 1/2$ then (4.3) and (4.4) imply that the supremum in (4.2) is $\mu_n(E)$. In order to estimate this measure we shall use BA $\{\zeta_n, n \in \mathbf{N}\}$ such that

$$\zeta_0 = T\eta_0 \text{ where } \eta_0 \sim \text{Bernoulli } 1/2 \text{ product measure and } T \text{ is defined by (1.3)} \quad (4.5)$$

From the definition of ζ_0 above and Assertion 1.1, we have that

$$\mu_n(E) = \mathbf{P}[\zeta_n(i) \neq 0 \text{ for some } i \in \{1, 2, \dots, k\}]$$

Let $L = L(n)$ ($R = R(n)$) denote the event that (in the considered BA) a particle that originated from $\{-n, -n+1, \dots, -n+k\}$ (resp., $\{n, n+1, \dots, n+k\}$) is in the region $K' := \{1, 2, \dots, k\}$ at time n . Obviously,

$$\mathbf{P}[L] \leq \mu_n(E) = \mathbf{P}[L \cap R^c] + \mathbf{P}[L^c \cap R] + \mathbf{P}[L \cap R] \leq \mathbf{P}[L] + \mathbf{P}[R] = 2\mathbf{P}[L]$$

whereas to derive the last inequality we used that changing ζ_0 to $-\zeta_0$ does not affect the distribution of ζ_0 defined in (4.5). As for $\mathbf{P}[L]$, one easily gets the following quite rough bounds

$$cP \leq \mathbf{P}[L] \leq ckP, \text{ for an appropriate } 0 < c < \infty$$

where P denotes the probability to have a positive particle at site 0 at time 0 and for this particle not to have survived till time n (observe, this is the same value, if 0 is substituted by any site of \mathbf{Z}).

To estimate the value of P we will use the following assertions. Their proofs are straightforward.

Assertion 4.1. *A particle in the BA will be annihilated till time n if and only if the distance between its initial position and that of its annihilating companion is $\leq 2n - 1$.*

Assertion 4.2. *Let ζ_0 be a configuration from $\{-1, 0, 1\}^{\mathbf{Z}}$. Assume $\zeta_0(i) = 1$ for some i . Let $f(\cdot)$ be the integrated profile corresponding to ζ_0 and such that $f(i) = 0$. Then $\min\{k \geq i + 2 : f(k) = 0\} - 1$ is the initial position of the annihilating companion in BA $\{\zeta_n, n \in \mathbf{N}\}$ of the particle that started from i .*

Let $f(\cdot)$ be the integrated process related to ζ_0 whose distribution is (4.5) conditioned to having a positive particle at 0. Then due to Assertions 4.1 and 4.2, we conclude that

$$P = \mathbf{P}[f(0) = 0, f(1) > 0, \dots, f(2n - 1) > 0 \mid f(1) > 0] \quad (4.6)$$

There are several ways to get the decay rate of (4.6). The promptest one goes along the following lines: The random variables $\{\zeta_0(i)\}_{i \in \mathbf{Z}}$ are not independent, but fall under the conditions of Chapter 4 of [Bi], which provide that $cn^{-1/2}f(n \cdot) \rightarrow B(\cdot)$ for some constant c . From this one concludes that (4.6) $\sim \text{const} \times n^{-1/2}$. This completes the proof. ♣

Remark 4.2. The applicability of the presented proof to another distribution of ζ_0 (and thus, also of η_0) depends basically on ones ability to calculate the asymptotics of (4.6) for this distribution.

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